$\boxed{\text{Real Analysis I}}$ Examination I (10/22/2008)

Total Score : 130

(1) Let $\{x_n\}$ be a bounded sequence of real numbers.

(i) (5%) State the following definitions respectively : a cluster point of $\{x_n\}$, $\limsup x_n$, and $\liminf x_n$.

(ii) (5%) Show that $\limsup x_n$ and $\liminf x_n$ are the largest and smallest cluster points of the sequence $\{x_n\}$.

(2) (i) (5%) State the following definitions respectively : the exterior measure, measurable sets, the Lebesgue measure, and σ-algebra.

(ii) (10%) Let

$$\mathcal{M} = \{E \subset \mathbf{R}^d : E \text{ is a measurable set}\}.$$

Prove that $\mathcal{M}$ is a σ-algebra.

(iii) (5%) Suppose E is a measurable subset of $\mathbf{R}^d$. Prove that for any $\varepsilon > 0$, there exists a closed set F with $F \subset E$ and $m(E - F) < \varepsilon$.

(3) (i) (5%) State the following definitions respectively : the measurable function, characteristic function, step function, and simple function.

(ii) (15%) Suppose f is measurable on $\mathbf{R}^d$. Prove that there exists a sequence of step functions $\{\psi_k\}_{k=1}^{\infty}$ that converges pointwise to $f(x)$ for almost every x.

(4) (20%) State and prove the theorems related to Littlewood's three principles respectively (i.e., Theorem 3.4(iv), Egorov's and Lusin's Theorems in the textbook).

(5) (10%) Construct a non-measurable set in $[0, 1]$.

(6) (10%) Suppose $A \subset E \subset B$, where A and B are measurable sets of finite measure. Prove that if $m(A) = m(B)$, then E is measurable.

(7) Let $\mathcal{C} \subset [0, 1]$ be the Cantor set constructed in the textbook (removing the interior of the middle third from each remaining interval at each stage).

(i) (5%) Prove that $\mathcal{C}$ is totally disconnected, i.e., given two distinct points $x, y \in \mathcal{C}$, there is a point $z \notin \mathcal{C}$ that lies between x and y.

(ii) (5%) Prove that $\mathcal{C}$ is perfect, i.e., $\mathcal{C}$ is closed and has no isolated points.

(8) (The Borel-Cantelli lemma) Suppose $\{E_k\}_{k=1}^{\infty}$ is a countable family of measurable subsets of $\mathbf{R}^d$ and that

$$\sum_{k=1}^{\infty} m(E_k) < \infty.$$

Let

$$E = \{x \in \mathbf{R}^d : x \in E_k \text{ for infinitely many } k\}.$$

(i) (5%) Show that E is measurable, and

$$E = \bigcap_{n=1}^{\infty} \bigcup_{k \geq n} E_k.$$

(ii) (5%) Prove $m(E) = 0$.

(9) Suppose E is a given set, and $\mathcal{O}_n$ is the open set defined by

$$\mathcal{O}_n = \{x : d(x, E) < 1/n\}.$$

(i) (5%) Show that if E is compact, then $m(E) = \lim_{n\to\infty} m(\mathcal{O}_n)$.

(ii) (5%) Does the conclusion in (i) also hold for E closed but unbounded? Explain your answer.

(10) (10%) Prove that $(a+b)^\gamma \geq a^\gamma + b^\gamma$, where $\gamma \geq 1$ and $a, b \geq 0$. Also, show that the reverse inequality holds when $0 \leq \gamma \leq 1$.